CALCULUS PRIMER

for

M.S. Students in Ag. Economics

This primer gives you some basic rules of derivatives, differentiation, optimization, and constrained optimization. Examples are provided, as well as some problems for you to work through. The keys to the problems are provided in this web site.

Part 1. Rate of change and derivatives

Suppose that Y=f(X) or Y is a function of X. One way of writing a change in Y with a change in X is $\Delta Y / \Delta X$. Now let the change in X go to a very small value, this becomes dY / dX (this is called a derivative).

How does Y change with a change in X (what is the *slope* of Y with respect to X)?

Let:

$Y=kX^n$

where:
k=constant (for example, 3)
n=power to which X is taken (for example, 2)

then:

$dY / dX = n * k * X^{n-1}$ (This is called the power function rule and is used for finding the derivative when X is taken to some power).

Example 1:

$Y=3X^2$

So k=3, n=2

$dY / dX = 3*2*X^{(2-1)} = 6*X$

Example 2:

Y=3 + 4X^2 + 2X^3

dY / dX =8X + 6X^2 (Note that '3' does not appear in the derivative, because it is a constant and the term does not contain X).

Example Problems to Work

Find the derivative of Y with respect to x:

1.1) **Y=8x**
1.2) **Y=3X^{-1}**
1.3) **Y=100x-6x^2**
1.4) **Y=-100x + 12x^2 - .75x^3**
1.5) **Y= 5*X$^{.5}$**
1.6) **Y=1/x$^{.33}$**
1.7) **Y=e^{-x}**
1.8) **Y=ln X**

Some helpful rules for more complex functions:

Sum-difference Rule

The derivative of a sum or difference of two functions is the sum or difference of the derivatives of the two functions.

$$\frac{d}{dX}[f(X)+g(X)] = \frac{d}{dX} f(x) + \frac{d}{dX} g(x)$$

example:

Y=3X^2 + 5X^3

$$\frac{dY}{dX} = \frac{d}{dX}3X^2 + \frac{d}{dX}5X^3 = 6X + 15X^2$$

Product Rule

The derivative of the product of two differentiable functions=first function derivative of the second function + second function*derivative of the first function.*

$$\frac{d}{dX}\,[f(X)*\,g(X)] = f(x)*\frac{d}{dX}g(x)+\frac{d}{dX}f(x)*g(x)$$

Example:

$Y=3X^2 * 5X^3$

$dY/dX = 3X^2 * 15X^2 + 6X * 5X^3 = 45X^4 + 30X^4 = 75X^4$

Quotient Rule

The derivative of the quotient of two functions f(x)/g(x) is

$$\frac{d}{dX}\,[f(X)/g(X)] = \frac{g(x)*\frac{d}{dX}f(x)-\frac{d}{dX}g(x)*f(x)}{g^2(x)}$$

Example:

$Y=3X^2 /5X^3$

$$\frac{dY}{dX}=\frac{6X*5X^3-3X^2 15X^2}{(5X^3)^2}=\frac{30X^4-45X^4}{25X^6}=\frac{-15X^4}{25X^6}=\frac{-3}{5X^2}$$

Chain Rule

If Z=f(Y) and Y=g(x), then

$$\frac{dZ}{dX} = \frac{dZ}{dY} * \frac{dY}{dX}.$$

Example:

Suppose $Z=3Y^2$ and Y=4X, find *dZ/dX*.

dZ/dX= 6*Y**4,

but since *Y*=4*X*, then

dZ/dX=6*4*X**4=*96X*

Example Problems to Work

Find the derivative of Y with respect to x:

1.9) $Y=8x*9x^2$
1.10) $Y=4X/3X^2$
1.11) $Y=100x^3-6x^2$
1.12) $Y=-10Z^2$ **and Z=7X**

Part 2. Partial derivatives

What if Y is a function of two variables X and Z?

Y=f(X,Z)

How does Y change with a change in X or Z?

If we look at a change in Y with a change in X, then we assume that Z is held constant. Also, if we look at a change in Y with a change in Z, then we assume that X is held constant.

Partial derivatives of Y with respect to X and Z:

$\partial Y / \partial X$ =change in Y with a change in X| Z held constant
$\partial Y / \partial Z$ =change in Y with a change in Z| X held constant

Example:

Let $Y=4X^2 + X*Z + 2Z^2$

How does Y change with a change in X?

$\partial Y / \partial X = 8X + Z$

(Notice that in this case, nothing is done with the $2Z^2$ term, because X is not found in that term. Hence it is treated as a constant)

How does Y change with change in Z? (Remember X is treated as a constant)

$\partial Y / \partial Z = X + 4Z$

(Notice that in this case, nothing is done with the $4X^2$ term, because Z is not found in that term. Hence it is treated as a constant)

Using the product rule

Example:

$Y=(5Z^2 + 3X)*(4Z+5X)$

$\partial Y/\partial Z = (5Z^2 + 3X)*4 + 10Z*(4Z+5X)=20Z^2 +12X + 40Z^2 +50ZX=60Z^2 + 12X +50ZX$

$\partial Y/\partial X = (5Z^2 + 3X)*5 + 3*(4Z+5X)=25Z^2+15X + 12Z +15X=25Z^2+30X + 12Z$

Using the quotient rule

$Y=(5Z^2 + 3X)/(4Z+5X)$

$\partial Y/\partial Z = [10Z*(4Z+5X)-(5Z^2 + 3X)*4]/ (4Z+5X)^2$
$= (40Z^2 +50ZX -20Z^2-12X)/ (4Z+5X)^2$
$= (20Z^2 - 12X+50ZX)/(4Z+5X)^2$

$\partial Y/\partial X =[3*(4Z+5X)-(5Z^2 + 3X)*5]/ (4Z+5X)^2= [12Z +15X -25Z^2-15X]/(4Z+5X)^2$
$=[-25Z^2+ 12Z]/(4Z+5X)^2$

Example Problems to Work

Find the partial derivative of Y with respect to X and Z:

2.1) $Y=X + X*Z - Z^2$

2.2) $Y=40 + 3X + 4Z^2$

2.3) $Y=6X^3 + 4*X*Z +Z^2$

2.4) $Y=(4X + 10Z)*(.5X + 2Z)$

2.5) $Y= (4X + 10Z)/(.5X + 2Z)$

Partial derivatives and elasticity

A partial derivative $\partial Y/\partial Z$ represents a slope or unit change over unit change. An elasticity is a percent change over percent change.

$E=(\partial Y/Y)/(\partial Z/Z)= \partial Y/\partial Z *(Z/Y)$=partial derivative of Y with respect to Z *(Z/Y).

Part 3. Differentials and taking the total differential

Differentials

The symbols *dY* and *dX* (infinitesimal changes in Y and X) are known as differentials. (Recall that *dY/dX* is known as the derivative). Sometimes, it may be useful to know what *dY* is for example, what would dY be given a particular change in X or *dX*.

Since *dY/dX=dY/dX*, we can rewrite it as

*dY=(dY/dX) *dX*

to solve for the differential *dY*

Example 1:

Recall example 1 in Part 1 of this primer. The function is:

$Y=3x^2$

So the derivative $dY / dX = 3*2*X^{(2-1)}=6*X$

What will the differential *dY* equal?

$dY= (dY / dX)\, dX =6*x*dX$

Suppose that the change in X is from 3 to 3.5, then we can solve for the change in Y, which would be 6*3*.5=12.

Example 2:

$Y=3 + 4X^2 + 2X^3$

$dY / dX =8X + 6X^2$

So $dY = (8X + 6X^2)\, dX$.

What if X changes from 1 to 2? Then the change in Y or *dY* is

(8*1 + 6*1*1) *1 = 14

What is the total differential?

Suppose that Y is a function of X and Z. The total differential is the total change in Y if X and Z change.

$$dY = (\partial Y / \partial X) * dX + (\partial Y / \partial Z) * dZ$$

Example 1:

Let $Y=4X^2 + X*Z + 2Z^2$

$\partial Y / \partial X = 8X + Z$
$\partial Y / \partial Z = X + 4Z$

dY=(8X + Z)* *dX* + (X+4Z)* *dZ*

Let X change from 2 to 3 and Z change from 4 to 5.

dY=(8*2+4)*1 + (2+4*4)*1=20 + 16 = 36

Example Problems to Work

Find the total differential:

3.1) **$Y=X + X*Z - Z^2$**

3.2) **$Y=40 + 3X + 4XZ^2$**

3.3) **For 3.1), solve for the numerical value of dY if X is 10 and changes to 11 and Z is 4 and changes to 4.5.**

Part 4. Optimization

So far, we have looked at a change in Y with a change in another variable, such as X, given that Y=f(X). Another value of interest may be where the function, f(x) is at its minimum or maximum point. At a minimum or maximum point, the slope will be ZERO.

If $\partial Y/\partial X$ is (-) then the function has a downward slope in Y with respect to X

If $\partial Y/\partial X$ is (+) then the function has upward slope in Y with respect to X

If $\partial Y/\partial X$ is (0) then the function has a flat slope in Y with respect to X

Even if the slope is zero, you don't know if you are at a maximum or a minimum. How can tell if you have a maximum or minimum? You must take the second derivative $\partial^2 Y/\partial X^2$ **.** The second derivative is found by taking the derivative of the first derivative with respect to X.

At a maximum

First order condition	Second order condition
$\partial Y/\partial X = 0$	$\partial^2 Y/\partial X^2 < 0$

At a minimum

First order condition	Second order condition
$\partial Y/\partial X = 0$	$\partial^2 Y/\partial X^2 > 0$

Example:

$Y=40X - 10X^2$
$\partial Y/\partial X = 40 - 20X$ (first derivative)

Set slope=0 and solve for X

0=40-20X
20X=40
X*=2

Is this a max or min?

$\partial^2 Y/\partial X^2 = \partial (40 - 20X)/ \partial X = -20$ (second derivative)

Since $\partial^2 Y/\partial X^2 < 0$, this is a maximum.

X^*=the value of X where Y is at a maximum

What if there is more than one variable in the function?

The partial derivatives of Y are taken with respect to each variable (X and Z) and set to zero to solve for the values for X* and Z*. (Recall that with the partial derivative, you are assuming that the other variable is held constant.)

Example:

$Y=40X - 10X^2 + 10Z – 20Z^2$

$\partial Y/\partial X = 40 - 20X = 0$
$\partial Y/\partial Z = 10 – 40Z = 0$

$X^*=2$
$Z^*=.25$

***In classes, checking the second order derivatives will be discussed in more detail for this situation.**

Example Problems to Work

Solve for the optimal values (where the slope is zero and then determine whether the value is a maximum or a minimum)

4.1) **Cost=8L^2-L (solve for L where ∂Cost/∂L=0 and check the second order derivative for max or min)**

4.2) **Profit=-Q^3 + 57 Q^2 –315Q-2000 (solve for Q where ∂Profit/∂Q=0 and check the second order derivative for max or min).**

Hint on 4.2) use quadratic formula

For ax^2 + bx+c=0
To find x values:

$$\frac{-b \pm (b^2 - 4ac)^{.5}}{2a}$$

Constrained Optimization

In the previous section, we looked at finding a maximum or minimum (optimization). Sometimes, optimization may be constrained by some factor. This is called constrained optimization.

Let:
Y=f(X,Z) function to be optimized (maximized or minimized)
C^0=c(X,Z) constraint placed on function

The Lagrangean is:

$L=f(X,Z) + \lambda(C^0-c(X,Z))$

L=Lagrangean
λ=Lagrangean multiplier
C^0 = fixed or limiting value

First order conditions for optimization are:

∂L/∂X=0
∂L/∂X =0
∂L/∂ë =0

Example:

Y=30X + 10Z +5XZ
Subject to:
40=X+Z

L=30X + 10Z +5XZ + ë(40-X-Z)

First order conditions
∂L/∂X=30 + 5Z-ë=0
∂L/∂X =10+5X- ë=0
∂L/∂ë =40-X-Z=0

Using the first two first order conditions:

X=4+Z

Substituting into the last first order condition

40=(4+Z) + Z
Z*=18
X*=22

Example Problems to Work

4.3) $Y=50X + 25Z + 5XZ$

Subject to:
$65=X+Z$