

MATH PRIMER

For

M.S. Students in Ag. Economics

This primer gives you a refresher on some basic algebra concepts, such as solving simultaneous equations, use of exponents and logs, and exponential and logarithmic functions.

SOLUTION OF SIMULTANEOUS EQUATIONS

If we know two ways that two variables are related, we can solve for the *equilibrium* value of those variables. For instance, suppose that price p and quantity Q are by a demand equation

$$Q_d = a - bP$$

and a supply equation

$$Q_s = c + dP.$$

We can solve these equations simultaneously to find the values for p and Q where demand equals supply at equilibrium. One way of solving simultaneous equations is to use the first equation to solve for one variable and substitute it into the second equations. For example, we can substitute the expression for Q in the first equation, $Q = a - bP$, into the second to get:

$$a - bP = c + dP.$$

Now we can solve for P by collecting terms:

$$\begin{aligned} a - c &= dP + bP, \text{ or} \\ a - c &= P(d + b). \end{aligned}$$

Dividing through by $(d + b)$, we get our equilibrium value of P :

$$P^* = \frac{a - c}{d + b}.$$

Now we can substitute this expression for P^* back into either the demand or supply equation to find the equilibrium expression for Q^* . For example, subbing P^* into the supply equation gives us:

$$\begin{aligned}
Q^* &= c + d\left(\frac{a-c}{d+b}\right) \\
&= \frac{c(d+b)}{d+b} + d\left(\frac{a-c}{d+b}\right) \\
&= \frac{cd + bc + ad - cd}{d+b} \\
&= \frac{bc + ad}{d+b}
\end{aligned}$$

So we now have equilibrium expressions for P^* and Q^* in terms of only the parameters a , b , c , and d . This is the point where the demand curve intersects the supply curve.

PROBLEMS:

1. Find the equilibrium solution for each of the following economic models of supply and demand
 - (a) $Q_d = 24 - 2P$, $Q_s = -5 + 7P$
 - (b) $Q_d = 51 - 3P$, $Q_s = 6P - 10$
 - (c) $Q_d = 30 - 2P$, $Q_s = -6 + 5P$

EXPONENTS AND LOGS, AND EXPONENTIAL AND LOGARITHMIC FUNCTIONS

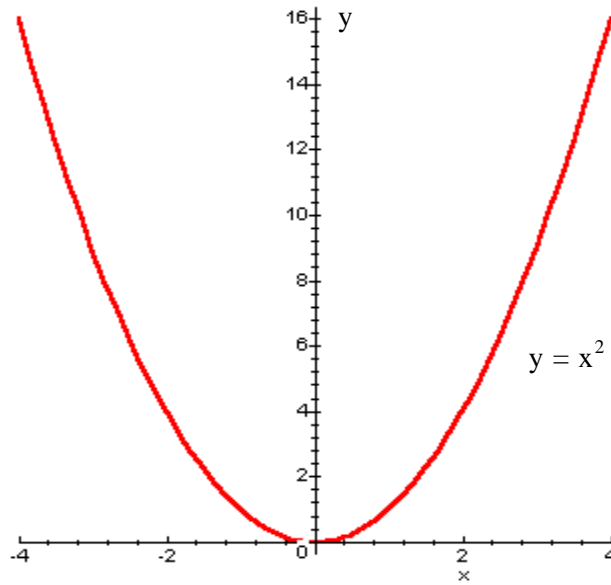
Exponents: The term *exponent* indicates the power to which a number or variable is raised. For example, in the expression x^3 , which is read “ x to the third power,” the exponent is three.

Numerical examples:

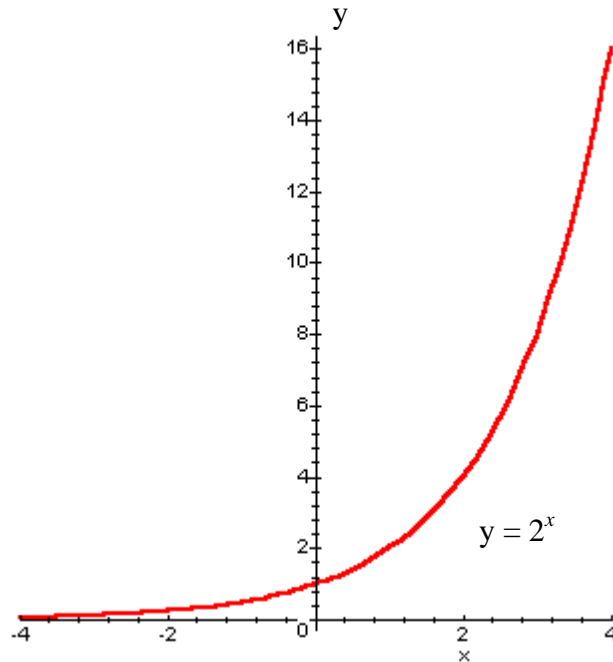
1. $3^2 = 3*3 = 9$
2. $2^5 = 2*2*2*2*2 = 32$

Graphical examples:

1. $y = x^2$



2. $y = 2^x$



Logarithms: Exponents are closely related to logarithms. Consider two numbers such as 3 and 9 that can be related by the equation $3^2 = 9$. Here we can define the *exponent* 2 to be the *logarithm* of 9 to the base of 3. We can write this new relationship as follows:

$$\log_3 9 = 2$$

The *base* of the logarithm, which is 3 in the example above, does not have to be restricted to any particular number. But in many applications, two numbers are widely chosen as bases – the number 10 and the *transcendental* number e . Logarithms that have a base equal to 10 are called *common logarithms*, or common “logs” for short.

Examples of common logs include:

$$\log_{10} 1000 = 3$$

$$\log_{10} 100 = 2$$

$$\log_{10} 10 = 1$$

$$\log_{10} 1 = 0$$

$$\log_{10} 0.1 = -1$$

$$\log_{10} 0.01 = -2$$

$$\log_{10} 5 = 0.68987$$

$$\log_{10} 2 = 0.30103$$

Logarithms that have a base equal to e are called *natural logarithms*, or natural “logs” for short. Natural logs even have a special notation – \ln instead of \log_e :

$$\text{Fore example, } \log_e 1 = \ln 1$$

Before proceeding with natural logs, we need to learn more about the number e .

The *transcendental number* e has many special properties, just like another famous transcendental number, π (pronounced *pi*). One way the number π is defined is by the relationship of the circumference (C) or area (A) of a circle to its radius (r):

$$\text{Circumference of a Circle: } C = 2\pi r \qquad \text{Area of a circle: } A = \pi r^2$$

One way the number e is defined is through the following function:

$$f(x) = \left(1 + \frac{1}{x}\right)^x.$$

For larger and larger values of x , the function $f(x)$ will get larger and larger, but by smaller amounts each time. For example, if we let x start at 1 and get larger, we find that

$$\begin{aligned} f(1) &= \left(1 + \frac{1}{1}\right)^1 = 2 \\ f(2) &= \left(1 + \frac{1}{2}\right)^2 = 2.25 \\ f(3) &= \left(1 + \frac{1}{3}\right)^3 = 2.37037\dots \\ f(4) &= \left(1 + \frac{1}{4}\right)^4 = 2.44141\dots \\ &\vdots \\ f(100) &= \left(1 + \frac{1}{100}\right)^{100} = 2.70481\dots \\ &\vdots \end{aligned}$$

The transcendental number e is defined as x approaches infinity (∞),

$$e = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x.$$

The numeric value of e is approximately 2.71828.

Examples of natural logarithms in include:

$$\begin{aligned} \ln e^3 &= \log_e e^3 = 3 \\ \ln e^2 &= \log_e e^2 = 2 \\ \ln e^1 &= \log_e e^1 = 1 \\ \ln 1 &= \log_e e^0 = 0 \\ \ln (1/e) &= \log_e e^{-1} = -1 \end{aligned}$$

Rules of Logarithms

Rule I (log of a product): $\ln(xy) = \ln x + \ln y$

Example 1: $\ln(e^7 e^2) = \ln e^7 + \ln e^2 = 7 + 2 = 9$

Example 2: $\ln(Be^5) = \ln B + \ln e^5 = \ln B + 5$

Coutner Example!: $\ln(u + v) \neq \ln u + \ln v$

Rule II (log of a quotient): $\ln(x/y) = \ln x - \ln y$

Example 3: $\ln(e^5/e^3) = \ln e^5 - \ln e^3 = 5 - 3 = 2$

Example 4: $\ln(e^4/c) = \ln e^4 - \ln c = 4 - \ln c$

Rule III (log of a power): $\ln(u^a) = a \ln u$

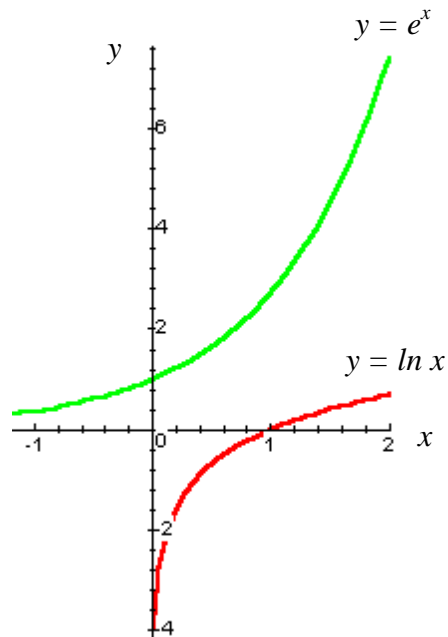
Example 3: $\ln(e^9) = 9 \ln e = 9$

Example 4: $\ln(a^3) = 3 \ln a$

Example 5: $\ln(uv^a) = \ln u + a \ln v$

Graphical Examples of log and exponential functions:

Graphs of $y = e^x$, and $y = \ln x$:



PROBLEMS:

1. Evaluate the following:

- (a) $\log_{10} 10,000$
- (b) $\log_{10} 0.01$
- (c) $\ln e^2$
- (d) $\ln(1/e^3)$
- (e) $\ln e^x - e^{\ln x}$

2. Evaluate the following by application of the rules of logarithms:

- (a) $\log_{10} (100)^{14}$
- (b) $\log_{10} (100)^{-1}$
- (c) $\ln (3/b)$
- (d) $\ln Ae^2$
- (e) $\ln Abe^{-4}$

3. Which of the following are valid?

- (a) $\ln u - 2 = \ln(u/e^2)$
- (b) $3 + \ln v = \ln(e^3/v)$
- (c) $\ln 3 + \ln 5 = 8$
- (d) $\ln u + \ln v - \ln w = \ln(uv/w)$

Economic Interpretation of the Transcendental Number e

Suppose we were lucky enough to find a bank willing to offer us the unusual interest rate of 100 percent, compounded continuously. Then the number $e = 2.71828$ can be interpreted as the year-end value to which \$1 will grow if interest at the rate of 100 percent per year were compounded continuously.

Compound interest formula:

What if we have an interest rate other than 100 percent? Suppose we have $r = 0.05$, or a 5 percent interest rate, and we start with principal of $\$A$, and invest it for t years. Finally, let's have m compounding period, so that if $m=1$, it's only compounded annually and if $m=12$, it's compounded monthly. The formula for the value, $V(m)$ of our investment is depends on m and the interest rate r :

$$V(m) = A \left(1 + \frac{r}{m} \right)^{mt} .$$

As the frequency of compounding increases, so does $V(m)$. To say that interest is compounded continuously is that same as saying that m approaches infinity in the limit. When this happens, our formula becomes:

$$V = \lim_{m \rightarrow \infty} V(m) = Ae^{rt} .$$

PROBLEM:

1. What's the value of an \$100 investment five years from now if the interest rate is $r = 0.06$ and
 - (a) Interest is compounded annually?
 - (b) Interest is compounded monthly?
 - (c) Interest is compounded continuously?

SOLVING QUADRATICS-THE QUADRATIC FORMULA

For a quadratic, such as

$$ax^2 + bx + c = 0$$

where a, b, and c are constants, the values for x may be found by using the quadratic formula:

$$\frac{-b \pm (b^2 - 4ac)^{.5}}{2a}$$

Example:

$$5X^2 + 10X - 15$$

$$a=5$$

$$b=10$$

$$c=-15$$

$$\frac{-10 \pm \sqrt{100 - 4(5)(-15)}}{2(5)} = \frac{10 \pm 20}{10} = 3 \quad \text{or} \quad -1$$